STABILITY OF AN EULERIAN ROD

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1. The stability of solutions of the nonlinear Euler equation

$$\frac{d^2y}{dx^2} + k^2\rho(x)\,y\left(1-\left(\frac{dy}{dx}\right)^2\right)^{0,5} = 0, \quad k^2 = \frac{Pl^2}{EI\pi^2} \tag{1.1}$$

was studied in [1] under the condition that the eigenvalues of the equation

$$\frac{d^2y}{dx^2} + k^2\rho(x)\,y = 0 \tag{1.2}$$

are simple. The analysis led to an entire class of problems for which the eigenvalues of Eq. (1.2) are doubly degenerate.

Continuing the analysis performed in [1], we now study Eq. (1.1) for $\rho(x) = x^\nu$, $-\infty < \nu < \infty$ and the boundary conditions

$$y(x_1) = 0, \quad dy(x_0)/dx = 0, \tag{1.3}$$

which are valid for the rod shown in Fig. 1. For this formulation of the problem, the total length of the rod is unknown, and for this reason the letter ℓ denotes the length corresponding to a dimensionless segment, bounded by the sections x_0 and x_1; the rest of the notation employed in Eqs. (1.1)-(1.3) is conventional or shown in Fig. 1.

The problem at hand has been solved only in a linear formulation [2]. The assumptions, made in this and a number of other works, that the rod, hinged at one end and subjected to a force P at the other end, is divided by the point where dy/dx = 0 into equal segments, and that the rod is symmetric with respect to this point, are valid only for rods with constant transverse sections $\nu = 0$.

In the general case $\nu \neq 0$ the point x_0 where dy/dx = 0 divides the rod into unequal segments, whose ratio is unknown and therefore can be determined from the solution. The law $\rho(x) = x^\nu$ must remain the same for the entire rod, otherwise the problem reduces to that studied in [1].

The stability of the solutions of the nonlinear equation (1.1) under the conditions (1.3) will be analyzed by the method of projections [3], taking into account the two-dimensionality of the null space of the operator

$$L_k = d^2/dx^2 + k^2x^\nu.$$

2. After the function $k^2x^\nu y\,(1-(dy/dx)^2)^{0.5}$ is expanded in a series in powers of y, dy/dx at the point (y, dy/dx) = (0, 0) Eq. (1.1) acquires the form

$$L_k y + \sum_{n=2}^{\infty} c_n y\,(dy/dx)^{n-1} = 0, \tag{2.1}$$

where

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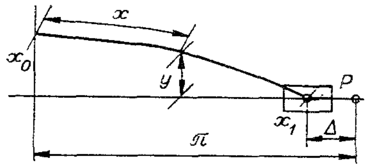

Fig. 1

$$c_n = \frac{1}{n!} \frac{\partial^n}{\partial y\, \partial (dy/dx)^{n-1}} \left(k^2 x^\nu y \left(1 - (dy/dx)^2\right)^{0,5}\right).$$

The fundamental system of solutions of the equation $L_k y = 0$ consists of the functions [4]

$$\varphi_{1i}(x) = \begin{cases} \sqrt{x}\, I_\omega (2\lambda_i \omega x^{1/2\omega}), & -\infty < \nu < -2,\ -2 < \nu < \infty, \\ \sqrt{x} \cos(\lambda_i \ln x), & \nu = -2, \end{cases}$$

$$\varphi_{2i}(x) = \begin{cases} \sqrt{x}\, N_\omega (2\lambda_i \omega x^{1/2\omega}), & -\infty < \nu < -2,\ -2 < \nu < \infty, \\ \sqrt{x} \sin(\lambda_i \ln x), & \nu = -2. \end{cases}$$

Here λ_i is the i-th eigenvalue of the operator L_k; $\omega = (\nu + 2)^{-1}$; $I_\omega(x)$ and $N_\omega(x)$ are Bessel functions of the first and second kinds of order ω.

For $-\infty < \nu < -2$, $-2 < \nu < \infty$ the eigenvalues λ_i ($i = 1, 2, \ldots$) are solutions of the equation

$$\det |a_{ij}| = 0,$$

where $a_{11} = \sqrt{x_1} I_\omega (2\omega\lambda x_1^{1/2\omega})$; $a_{12} = \sqrt{x_1} N_\omega (2\omega\lambda x_1^{1/2\omega})$; $a_{21} = \frac{1}{\sqrt{x_0}} I_\omega (2\omega\lambda x_0^{1/2\omega}) - \lambda x_0^{(1-\omega)/2\omega} I_{\omega+1}(2\omega\lambda x_0^{1/2\omega})$; $a_{22} = \frac{1}{\sqrt{x_0}} N_\omega (2\omega\lambda x_0^{1/2\omega}) - \lambda x_0^{(1-\omega)/2\omega} N_{\omega+1}(2\omega\lambda x_0^{1/2\omega})$, and for $\nu = -2$ they are solutions of the equation $\tan(\lambda \ln(x_1/x_0)) = 2\lambda$.

The smallest eigenvalue λ_1 makes it possible to write the stability condition for the solution of the equation $L_k y = 0$ under the conditions (1.3) in the form

$$\begin{aligned} \mu &= k^2 - \lambda_1^2 \leqslant 0, \quad -\infty < \nu < -2,\ -2 < \nu < \infty, \\ \mu &= k^2 - \lambda_1^2 - 0{,}25 \leqslant 0, \quad \nu = -2. \end{aligned} \tag{2.2}$$

The last condition (2.2) is well known [2]. The equal sign in Eq. (2.2) corresponds to the stability boundary and determines the critical value k_*^2, obtained in the linear approximation. By virtue of the relations (2.2) the operator L_k can be formally replaced by the operator $L_\mu = d^2/dx^2 + (\mu + \lambda_1^2) x^\nu$, which leads to simpler formulas.

Before solving Eq. (1.1), we shall show that the space of the functions $\varphi_{1i}(x)$, $\varphi_{2i}(x)$ ($i = 1, 2, \ldots$), which are square-integrable on the interval (x_0, x_1), is a Hilbert space with the scalar product $\langle \varphi_{ki}(x), \overset{*}{\varphi}_{nj}(x) \rangle$ ($\overset{*}{\varphi}_{nj}(x)$ is the function conjugate to the function $\varphi_{nj}(x)$ with respect to the scalar product).

Since the functions $\varphi_{kj}(x)$ ($k = 1, 2$, $j = 1, 2, \ldots$) are orthogonal on the interval (x_0, x_1) with x^ν as the weighting function, $\overset{*}{\varphi}_{kj}(x) = x^\nu \varphi_{kj}(x)$, to within constant factors, over the entire interval $-\infty < \nu < \infty$.

Now, having determined the amplitude as the projection of the function y(x) on the characteristic subspace associated with the conjugate vectors $\overset{*}{\varphi}_{11}(x)$, $\overset{*}{\varphi}_{21}(x)$, $\varepsilon = \langle (y(x), y(x)), (\overset{*}{\varphi}_{11}(x), \overset{*}{\varphi}_{21}(x)) \rangle$, we seek the solution of Eq. (2.1) in the form of the series expansions

$$y(x) = \sum_{n=1}^{\infty} \frac{1}{n!} \varepsilon^n y_n(x), \quad \mu_n = \sum_{n=1}^{\infty} \frac{1}{n!} \varepsilon^n \mu_n. \tag{2.3}$$

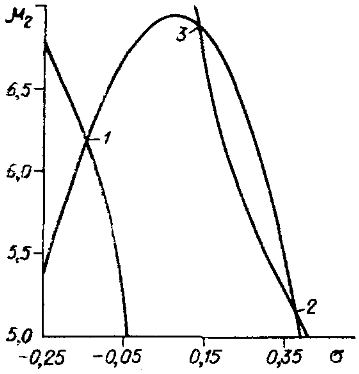

Fig. 2

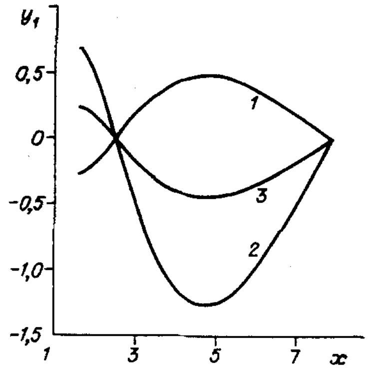

Fig. 3

After substituting the series (2.3) into Eq. (2.1) and equating terms with like powers of ε up to third order inclusively we arrive at the system of equations

$$L_0 y_1 = 0; \tag{2.4}$$

$$L_0 y_2 + 2\mu_1 \frac{\partial L_0}{\partial \mu} y_1 + 2\mathbf{B}(\mathbf{y}_1, \mathbf{y}_1) = 0; \tag{2.5}$$

$$L_0 y_3 + 3\mu_1 \frac{\partial L_0}{\partial \mu} y_2 + 6\mathbf{B}(\mathbf{y}_1, \mathbf{y}_2) + 3\mu_2 \frac{\partial L_0}{\partial \mu} y_1 + 6\mathbf{C}(\mathbf{y}_1, \mathbf{y}_1, \mathbf{y}_1) = 0, \tag{2.6}$$

where $\mathbf{B}(\mathbf{y}_1, \mathbf{y}_2)$ and $\mathbf{C}(\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3)$ are matrix differential operators, defined as

$$\mathbf{B}(\mathbf{y}_1, \mathbf{y}_2) = \frac{c_2\lambda_1^2}{2}\left(y_1 \frac{dy_2}{dx} + y_2 \frac{dy_1}{dx}\right),$$

$$\mathbf{C}(\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3) = \frac{c_3\lambda_1^2}{3}\left(y_1 \frac{dy_2}{dx}\frac{dy_3}{dx} + y_2 \frac{dy_1}{dx}\frac{dy_3}{dx} + y_3 \frac{dy_1}{dx}\frac{dy_2}{dx}\right).$$

The solution of Eq. (2.4) is a linear combination of the functions $\varphi_{11}(x), \varphi_{21}(x)$, $y_1 = \varphi_{11}(x) + \sigma\varphi_{21}(x)$, where σ is a factor.

Equations (2.5) and (2.6) have a solution when, and only when, for k = 1 and 2

$$\langle L_0 y_2, \varphi_{k1}^*(x)\rangle = \langle L_0 y_3, \varphi_{k1}^*(x)\rangle = 0,$$

which follow from Fredholm's theorem of the alternative. Taking this into account and forming the scalar product of Eqs. (2.5) and (2.6) by $\varphi_{11}^*(x)$, $\varphi_{21}^*(x)$ we obtain

$$\mu_1 \left\langle \frac{\partial L_0}{\partial \mu} y_1, \varphi_{k1}^*(x)\right\rangle + \langle \mathbf{B}(\mathbf{y}_1, \mathbf{y}_1), \varphi_{k1}^*(x)\rangle = 0,$$

$$\mu_1 \left\langle \frac{\partial L_0}{\partial \mu} y_2, \varphi_{k1}^*(x)\right\rangle + 2\langle \mathbf{B}(\mathbf{y}_1, \mathbf{y}_2), \varphi_{k1}^*(x)\rangle + \mu_2 \left\langle \frac{\partial L_0}{\partial \mu} y_1, \varphi_{k1}^*(x)\right\rangle +$$
$$+ 2\langle \mathbf{C}(\mathbf{y}_1, \mathbf{y}_1, \mathbf{y}_1), \varphi_{k1}^*(x)\rangle = 0, \quad k = 1, 2,$$

whence, since $c_2 = 0$ and therefore $\mu_1 = 0$ and $y_2 = 0$, we find

$$\begin{aligned} &\mu_2 \left\langle \frac{\partial L_0}{\partial \mu} y_1, \varphi_{11}^*(x)\right\rangle + 2\langle \mathbf{C}(\mathbf{y}_1, \mathbf{y}_1, \mathbf{y}_1), \varphi_{11}^*(x)\rangle = 0, \\ &\mu_2 \left\langle \frac{\partial L_0}{\partial \mu} y_1, \varphi_{21}^*(x)\right\rangle + 2\langle \mathbf{C}(\mathbf{y}_1, \mathbf{y}_1, \mathbf{y}_1), \varphi_{21}^*(x)\rangle = 0. \end{aligned} \tag{2.7}$$

After substituting the expressions for y_1 into Eqs. (2.7) we arrive at a system of two algebraic equations

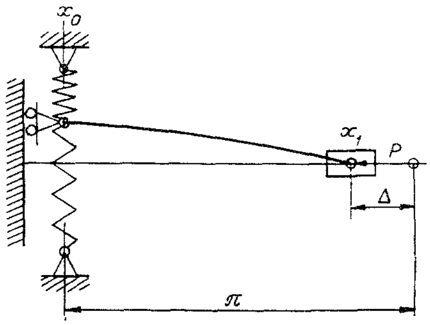

Fig. 4

$$C_{13}\sigma^3 + C_{12}\sigma^2 + C_{11}\sigma + B_{12}\sigma\mu_2 + B_{11}\mu_2 + B_{10} = 0,$$
$$C_{23}\sigma^3 + C_{22}\sigma^2 + C_{21}\sigma + B_{22}\sigma\mu_2 + B_{21}\mu_2 + B_{20} = 0, \tag{2.8}$$

where

$$C_{13} = -\lambda_1^2\langle x^\nu\varphi_{21}(x)(d\varphi_{21}(x)/dx)^2,\ \varphi_{11}^*(x)\rangle;$$
$$C_{12} = -\lambda_1^2\langle x^\nu(\varphi_{11}(x)(d\varphi_{21}(x)/dx)^2 + 2\varphi_{21}(x)\,d\varphi_{11}(x)/dx\,d\varphi_{21}(x)/dx),\ \varphi_{11}^*(x)\rangle;$$
$$C_{11} = -\lambda_1^2\langle x^\nu(2\varphi_{11}(x)\,d\varphi_{11}(x)/dx\,d\varphi_{21}(x)/dx + \varphi_{21}(x)(d\varphi_{11}(x)/dx)^2),\ \varphi_{11}^*(x)\rangle;$$
$$B_{12} = \langle x^\nu\varphi_{21}(x),\ \varphi_{11}^*(x)\rangle;$$
$$B_{11} = \langle x^\nu\varphi_{11}(x),\ \varphi_{11}^*(x)\rangle;$$
$$B_{10} = -\lambda_1^2\langle x^\nu\varphi_{11}(x)(d\varphi_{11}(x)/dx)^2,\ \varphi_{11}^*(x)\rangle;$$
$$C_{23} = -\lambda_1^2\langle x^\nu\varphi_{21}(x)(d\varphi_{21}(x)/dx)^2,\ \varphi_{21}^*(x)\rangle;$$
$$C_{22} = -\lambda_1^2\langle x^\nu(\varphi_{11}(x)(d\varphi_{21}(x)/dx)^2 + 2\varphi_{21}(x)\,d\varphi_{11}(x)/dx\,d\varphi_{21}(x)/dx),\ \varphi_{21}^*(x)\rangle;$$
$$C_{21} = -\lambda_1^2\langle x^\nu(2\varphi_{11}(x)\,d\varphi_{11}(x)/dx\,d\varphi_{21}(x)/dx + \varphi_{21}(x)(d\varphi_{11}(x)/dx)^2),\ \varphi_{21}^*(x)\rangle;$$
$$B_{22} = \langle x^\nu\varphi_{21}(x),\ \varphi_{21}^*(x)\rangle;$$
$$B_{21} = \langle x^\nu\varphi_{11}(x),\ \varphi_{21}^*(x)\rangle;$$
$$B_{20} = -\lambda_1^2\langle x^\nu\varphi_{11}(x)(d\varphi_{11}(x)/dx)^2,\ \varphi_{21}^*(x)\rangle.$$

Following the usual procedure of calculating the stability [3], we determine the quadratic form of the system (2.8), using the method of successive approximations. As a first approximation we can take the solution of the linearized system (2.8):

$$\sigma_1 = \frac{B_{21}B_{10} - B_{11}B_{20}}{B_{11}C_{21} - B_{21}C_{11}}.$$

Substituting the expressions for σ_1 into Eq. (2.8) gives an equation for two conical sections in the (μ_2, σ) plane:

$$g_1(\mu_2, \sigma) = C_{12}\sigma^2 + C_{11}\sigma + B_{12}\sigma\mu_2 + B_{11}\mu_2 + \overline{B}_{10} + O|\sigma|^3 = 0,$$
$$g_2(\mu_2, \sigma) = C_{22}\sigma^2 + C_{21}\sigma + B_{22}\sigma\mu_2 + B_{21}\mu_2 + \overline{B}_{20} + O|\sigma|^3 = 0. \tag{2.9}$$

Here $\overline{B}_{10} = B_{10} + C_{13}\sigma_1^3$ and $\overline{B}_{20} = B_{20} + C_{23}\sigma_1^3$.

The existence of two independent variables μ_2 and σ guarantees the existence of solutions of the system (2.9). Depending on the sign of the discriminant of the cubic equation equivalent to the system (2.9)

$$G = G_3\sigma^3 + G_2\sigma^2 + G_1\sigma + G_0 = 0,$$

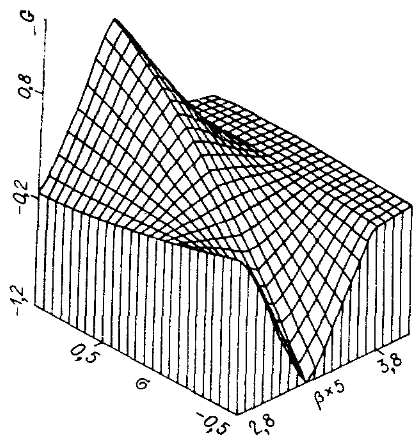

Fig. 5

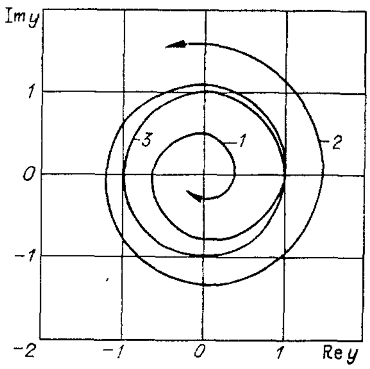

Fig. 6

where $G_3 = B_{12}C_{22} - B_{22}C_{12}$; $G_2 = B_{11}C_{22} + B_{12}C_{21} - B_{21}C_{12} - B_{22}C_{11}$; $G_1 = B_{11}C_{21} - B_{12}\bar{B}_{20} - B_{22}\bar{B}_{10} - B_{21}C_{11}$; $G_0 = B_{11}\bar{B}_{20} - B_{21}\bar{B}_{10}$, the system (2.9) has, besides the trivial solution y(x) = 0, three real solutions or a single real and two complex-conjugate solutions. If the discriminant is zero, then two or all three real roots are equal to one another. The existence of real and complex solutions is explained by the fact that under the action of the force P the rod will be deformed from the state y(x) = 0, $x_0 \le x \le x_1$ into a new state of equilibrium, stable or unstable, undergoing monotonic or oscillatory motion. This explanation becomes quite obvious, if the parameter μ is considered to be the time in the space of images after the corresponding initial-boundary-value problem with the initial condition y(x) = 0 is transformed into this space (for example, by a Laplace transform).

Another aspect of the existence of a set of solutions $(\mu_2^{(i)}, \sigma^{(i)})$ (i = 1-3) of the system (2.9) is that these solutions refer to different rods or to rods of different length, the other parameters being the same. Indeed, since λ_1 and ε are the same for all solutions $(\mu_2^{(i)}, \sigma^{(i)})$ (i = 1-3), it follows from Eqs. (2.2) and (2.3) that to each solution of the system (2.9) there corresponds the characteristic value $k^2 = \lambda_1^2 + 0.5\mu_2^{(i)}\varepsilon^2$ and, therefore, a length.

Calculations performed taking as an example the starting data $x_0 = 3\pi/2$ and $\nu = -2$ showed that the system (2.9) has three real solutions: $(\mu_2^{(1)}; \sigma^{(1)}) = (6.192; -0.158)$, $(\mu_2^{(2)}; \sigma^{(2)}) = (5.161; 0.375)$, $(\mu_2^{(3)}; \sigma^{(3)}) = (6.880; 0.133)$ (Fig. 2, points of intersection of the conical sections 1-3).

The deflection $y_1(x)$ in the range $x = x_0 \pm \pi$ was calculated for each of the solutions, presented above, of the system (2.9). The results of the calculations are shown in Fig. 3, where the numbers of the solutions of the system (2.9) correspond to the numbers of the rods. The computational results show that for all rods the deflection is zero at the point (hinge point) $x_2 = 0.787\pi$ ($y(x_1) = 0$ by definition), so that, as noted above, $|x_0 - x_2| \neq |x_1 - x_0|$.

Analysis of the stability of the solutions of Eq. (2.1) with the conditions (1.3) reduces to determining the eigenvalues of the Jacobian matrix of the system (2.9) with appropriate parameterization and should be done for each i-th rod using the values obtained for $(\mu_2^{(i)}, \sigma^{(i)})$. The parameterization of the system (2.9) consists of representing the functions $g_i(\mu_2, \sigma)$ (i = 1, 2) as functions of the parameter μ. Combining Eqs. (2.3) and (2.9) and using the normalization condition ε = 1 we can write the system (2.9) in the form

$$\bar{g}_1(\mu) = 4\mu^2 (C_{12}\sigma^2\mu_2^{-2} + C_{11}\sigma\mu_2^{-2} + B_{12}\sigma\mu_2^{-1} + B_{11}\mu_2^{-1} + \bar{B}_{10}\mu_2^{-2}) + O(|\sigma| + |\mu|)^3 = 0,$$

$$\bar{g}_2(\mu) = 4\mu^2 (C_{22}\sigma^2\mu_2^{-2} + C_{21}\sigma\mu_2^{-2} + B_{22}\sigma\mu_2^{-1} + B_{21}\mu_2^{-1} + \bar{B}_{20}\mu_2^{-2}) + O(|\sigma| + |\mu|)^3 = 0.$$

The the solution of Eq. (2.1), with the conditions (1.3), is stable, if the real parts of the eigenvalues of the Jacobian matrix

$$M = |\varepsilon_{ij}|,$$

where $\varepsilon_{11} = \partial \bar{g}_1(\mu)/\partial \mu_2^{-1}$; $\varepsilon_{12} = \partial \bar{g}_1(\mu)/\partial(\sigma\mu_2^{-1})$; $\varepsilon_{21} = \partial \bar{g}_2(\mu)/\partial \mu_2^{-1}$; $\varepsilon_{22} = \partial \bar{g}_2(\mu)/\partial(\sigma\mu_2^{-1})$, are negative. Since at each point $(\mu_2(n), \sigma^{(n)})$ with small μ we have $\det M = \mu^2 \det M(\mu_2^{(n)}, \sigma^{(n)}) + O|\mu|^3$, we write the stability conditions in the form

$$\max(\mu \operatorname{Re} s_1^{(n)}, \mu \operatorname{Re} s_2^{(n)}) < 0,$$
$$|\operatorname{Re}(\varepsilon_{11}^{(n)} + \varepsilon_{22}^{(n)})| > |(\alpha_n^2 + \beta_n^2)^{0,25} \cos(0,5 \operatorname{arctg} \alpha_n^{-1}\beta_n)|. \qquad (2.10)$$

Here $s_1^{(n)}$, $s_2^{(n)}$ are the eigenvalues of the matrix

$$M(\mu_2^{(n)}, \sigma^{(n)}),\ \varepsilon_{ij}^{(n)} = \varepsilon_{ij}(\mu_2^{(n)}, \sigma^{(n)}),$$
$$\alpha_n = (\operatorname{Re}(\varepsilon_{11}^{(n)} - \varepsilon_{22}^{(n)}))^2 - (\operatorname{Im}(\varepsilon_{11}^{(n)} - \varepsilon_{22}^{(n)}))^2 +$$
$$+ 4 \operatorname{Re}\varepsilon_{12}^{(n)} \operatorname{Re}\varepsilon_{21}^{(n)} - 4 \operatorname{Im}\varepsilon_{12}^{(n)} \operatorname{Im}\varepsilon_{21}^{(n)},$$
$$\beta_n = 2 \operatorname{Re}(\varepsilon_{11}^{(n)} - \varepsilon_{22}^{(n)}) \operatorname{Im}(\varepsilon_{11}^{(n)} - \varepsilon_{22}^{(n)}) + 4 \operatorname{Re}\varepsilon_{12}^{(n)} \operatorname{Im}\varepsilon_{21}^{(n)} + 4 \operatorname{Re}\varepsilon_{21}^{(n)} \operatorname{Im}\varepsilon_{12}^{(n)}.$$

If the eigenvalues $s_1^{(n)}$, $s_2^{(n)}$ are real, then the conditions (2.10) simplify:

$$\max(\mu s_1^{(n)}, \mu s_2^{(n)}) < 0, \quad \det M(\mu_2^{(n)}, \sigma^{(n)}) > 0. \qquad (2.11)$$

We note that the inequalities (2.11) are valid when, and only when, the conical sections (2.9) intersect strictly in the plane (μ_2, σ), i.e.,

$$\det \begin{vmatrix} \partial g_1(\mu_2, \sigma)/\partial \mu_2 & \partial g_1(\mu_2, \sigma)/\partial \sigma \\ \partial g_2(\mu_2, \sigma)/\partial \mu_2 & \partial g_2(\mu_2, \sigma)/\partial \sigma \end{vmatrix} \neq 0.$$

Returning to the complex values $s_1^{(n)}$, $s_2^{(n)}$, we note that a rod loaded with a constant force cannot be an auto-oscillatory system and therefore the real parts $\operatorname{Re} s_1^{(n)}$, $\operatorname{Re} s_2^{(n)}$, cannot change sign, passing through zero.

The calculation of the eigenvalues of the Jacobian matrix for the solutions obtained above $(\mu_2(n), \sigma^{(n)})$ (n = 1-3) gave the following results:

$$(s_1^{(1)}; s_2^{(1)}) = (-9,504 + 7,684i; -9,504 - 7,684i),$$
$$(s_1^{(2)}; s_2^{(2)}) = (2,852 + 10,200i; 2,852 - 10,200i),$$
$$(s_1^{(3)}; s_2^{(3)}) = (12,873; -6,005),$$

Whence it follows that the first rod is stable for $\mu > 0$, the second rod is stable for $\mu < 0$, and the third rod is unstable for any μ.

These results require some explanation. We start with the fact that, as one can see from Eq. (2.2), the value $\mu = 0$ corresponds to the critical force $P_* = P(k_*^2)$ such that $P > P_*$ for $\mu > 0$ and vice versa. Then stability of the second rod for $P < P_*$ corresponds to the general ideas about loaded rods; the condition of stability of the first rod $P > P_*$ does not mean that P can be arbitrarily large (the conditions (2.10) and (2.11) are valid for small values of μ), and indicates that for $P < P_*$ the stable position of the rod does not correspond to $dy(x_0)/dx = 0$. This condition will be achieved at a different point x or at all points x in the case of the trivial solution. As far as the third rod is concerned, its deformed state with $dy(x_0)/dx = 0$ is stable for any P, i.e., for a rod of fixed length and configuration it is impossible to achieve with a force P a stable deformed state for which maximum deflection would be achieved at the point x_0.

3. We now consider the rod shown in Fig. 4. Let the force P depend on the deflection:

$$P = P_0(1 - \beta\,\Delta(y)). \tag{3.1}$$

Here P_0 and β are constants:

$$\Delta(y) = \pi - \int_{x_0}^{x_1} (1 - (dy/dx)^2)^{0.5} dx.$$

When a force P is applied at x_1, the other end of the rod x_0 can shift in a direction perpendicular to the axis of the rod, overcoming the resistance of the spring with spring constant α. As the deflection increases, P decreases and the spring will cause the rod to return to its initial position. In this formulation, there exist all prerequisites for the appearance of undamped periodic oscillations – limit cycles.

After expanding in powers of (y, dy/dx) at the point (y, dy/dx) = (0, 0) Eq. (1.1) acquires the form

$$L_k y + \sum_{n=2}^{\infty} c_n y((dy/dx)^{n-1} - \beta \int_{x_0}^{x_1} (dy/dx)^{n-1} dx) = 0, \quad k^2 = \frac{P_0 l^2}{EI\pi^2}. \tag{3.2}$$

The boundary conditions corresponding to the manner in which the rod is attached are

$$y(x_1) = 0, \quad dy(x_0)/dx + \alpha y(x_0) = 0, \tag{3.3}$$

which lead to different (compared with Eq. (1.3)) relations for the eigenvalues of the operator L_k. Now they are solutions of the equation

$$\det |a_{ij}| = 0,$$

where

$$a_{11} = \sqrt{x_1}\, I_\omega(2\omega\lambda x_1^{1/2\omega}), \quad a_{12} = \sqrt{x_1}\, N_\omega(2\omega\lambda x_1^{1/2\omega});$$

$$a_{21} = \left(\frac{1}{\sqrt{x_0}} + \alpha \sqrt{x_0}\right) I_\omega(2\omega\lambda x_0^{1/2\omega}) - \lambda x_0^{(1-\omega)/2\omega} I_{\omega+1}(2\omega\lambda x_0^{1/2\omega});$$

$$a_{22} = \left(\frac{1}{\sqrt{x_0}} + \alpha \sqrt{x_0}\right) N_\omega(2\omega\lambda x_0^{1/2\omega}) - \lambda x_0^{(1-\omega)/2\omega} N_{\omega+1}(2\omega\lambda x_0^{1/2\omega}).$$

for $-\infty < \nu < -2$, $-2 < \nu < \infty$ and the equation

$$(1 + 2\alpha x_0)\,\mathrm{tg}(\lambda \ln(x_1/x_0)) = 2\lambda$$

with $\nu = -2$.

The rest of the procedure of analyzing the stability of the solutions (3.2) is similar to Sec. 2 for a rod loaded with a constant force. For this reason, using the notation

$$\Delta_i(y) = \int_{x_0}^{x_1} dy_i/dx\, dx, \quad \Delta_{ij}(y) = \int_{x_0}^{x_1} dy_i/dx\, dy_j/dx\, dx,$$

$$\Delta_{ij} = \int_{x_0}^{x_1} d\varphi_{i1}(x)/dx\, d\varphi_{j1}(x)/dx\, dx,$$

we indicate only the differences in the analysis. This concerns the matrix differential operators

$$\mathbf{B}(y_1, y_2) = \frac{c_2\lambda_1^2}{2}\left(y_1\left(\frac{dy_2}{dx} - \beta\Delta_2(y)\right) + y_2\left(\frac{dy_1}{dx} - \beta\Delta_1(y)\right)\right),$$

$$\mathbf{C}(y_1, y_2, y_3) = \frac{c_3\lambda_1^2}{3}\left(y_1\left(\frac{dy_2}{dx}\frac{dy_3}{dx} - \beta\Delta_{23}(y)\right) + \right.$$
$$\left. + y_2\left(\frac{dy_1}{dx}\frac{dy_3}{dx} - \beta\Delta_{13}(y)\right) + y_3\left(\frac{dy_1}{dx}\frac{dy_2}{dx} - \beta\Delta_{12}(y)\right)\right),$$

which lead to the coefficients

$$C_{i3} = \lambda_1^2\left(\beta\Delta_{22}\langle x^\nu\varphi_{21}(x),\ \varphi_{i1}^*(x)\rangle - \langle x^\nu\varphi_{21}(x)(d\varphi_{21}(x)/dx)^2,\ \varphi_{i1}^*(x)\rangle\right),$$
$$C_{i2} = \lambda_1^2(\beta\langle x^\nu(\Delta_{22}\varphi_{11}(x) + 2\Delta_{12}\varphi_{21}(x)),\ \varphi_{i1}^*(x)\rangle - \langle x^\nu(\varphi_{11}(x)(d\varphi_{21}(x)/dx)^2 +$$
$$+ 2\varphi_{21}(x)\,d\varphi_{11}(x)/dx\,d\varphi_{21}(x)/dx),\ \varphi_{i1}^*(x)\rangle),$$
$$C_{i1} = \lambda_1^2(\beta\langle x^\nu(2\Delta_{12}\varphi_{11}(x) + \Delta_{11}\varphi_{21}(x)),\ \varphi_{i1}^*(x)\rangle -$$
$$- \langle x^\nu(2\varphi_{11}(x)\,d\varphi_{11}(x)/dx\,d\varphi_{21}(x)/dx + \varphi_{21}(x)(d\varphi_{11}(x)/dx)^2),\ \varphi_{i1}^*(x)\rangle),$$
$$B_{i0} = \lambda_1^2\left(\beta\langle x^\nu\Delta_{11}\varphi_{11}(x),\ \varphi_{i1}^*(x)\rangle - \langle x^\nu\varphi_{11}(x)(d\varphi_{11}(x)/dx)^2,\ \varphi_{i1}^*(x)\rangle\right),\quad i = 1, 2.$$

The remaining coefficients in the system (2.9) are unchanged, though, of course, they depend implicitly on α through λ_1.

Continuing the stability analysis, just as in Sec. 2, we can find solutions of the equation $G = 0$. The function $G = G(\sigma, \beta)$, obtained with $\nu = 0$, $x_0 = 0$, and $\alpha = 1$, is presented in Fig. 5. Using the same initial data and fixed values of β the solutions of the system (2.9) and the eigenvalues of the Jacobian matrix which correspond to the solutions were determined. The results were as follows:

a) with $\beta = 0.550$

$$\left(\mu_2^{(1)};\ \sigma^{(1)}\right) = (-1.496;\ 0{,}123),\ \left(s_1^{(1)};\ s_2^{(1)}\right) =$$
$$= (2.536;\ -0.350),\ \left(\mu_2^{(2)};\ \sigma^{(2)}\right) = (-1{,}136 - 0{,}411i;\ 0{,}352 + 1{,}663i),$$
$$\left(s_1^{(2)};\ s_2^{(2)}\right) = (3{,}175 - 0{,}612i;\ 0{,}0439 + 0{,}285i)$$

[the third solution $(\mu_2^{(3)}, \sigma^{(3)})$, the complex conjugate of the second solution, is not presented];

b) with $\beta = 0.612$

$$\left(\mu_2^{(1)};\ \sigma^{(1)}\right) = (-1{,}759;\ 0{,}168),\ \left(s_1^{(1)};\ s_2^{(1)}\right) = (2{,}927;\ -0{,}269),$$
$$\left(\mu_2^{(2)};\ \sigma^{(2)}\right) = (-1{,}423 + 0{,}273i;\ 0{,}0856 - 0{,}998i),$$
$$\left(s_1^{(2)};\ s_2^{(2)}\right) = (3{,}483 + 0{,}410i;\ 0 - 0{,}252i);$$

c) with $\beta = 0.700$

$$\left(\mu_2^{(1)};\ \sigma^{(1)}\right) = (-2.256;\ 0.372),\quad \left(s_1^{(1)},\ s_2^{(1)}\right) = (3{,}431;\ -0{,}254),$$
$$\left(\mu_2^{(2)};\ \sigma^{(2)}\right) = (-1{,}887 + 0{,}0982i;\ -0{,}114 - 0.481i),$$
$$\left(s_1^{(2)};\ s_2^{(2)}\right) = (3{,}809 + 0{,}188i;\ -0{,}0164 - 0{,}272i).$$

According to the calculations, for all values of β indicated the real solutions corresponding to states of stationary equilibrium of the rod are unstable for any value of μ. The complex-conjugate solutions, corresponding to vibrational states of the rod, in the case a are stable for $\mu < 0$ (Fig. 6, curve 1), they are unstable in the case c for any μ, and for $\mu < 0$ the phase trajectory has the form of the curve 2. In the case b the vibrational regime is a limit cycle with $\mu < 0$ (curve 3). Here, just as in Sec. 2, in each of the cases a-c the different solutions of the system (2.9) correspond to rods of different length. For complex-conjugate solutions the rod lengths are the same.

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